A family of Hofstadter's recursive functions : more on G and beyond

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Hofstadter's functions G and H

From Douglas Hofstadter, "Gödel,Escher,Bach", chapter 5 :

$$G : \mathbb{N} \to \mathbb{N}$$
$$G(0) = 0$$
$$G(n) = n - G(G(n-1)) \quad \text{otherwise}$$

$$H : \mathbb{N} \to \mathbb{N}$$
$$H(0) = 0$$
$$H(n) = n - H(H(H(n-1))) \quad \text{otherwise}$$

In the On-Line Encyclopedia of Integer Sequences (OEIS):

A5206 and A5374

Beyond : a family F_k of functions

For any number k of nested recursive calls:

$$\begin{aligned} F_k &: \mathbb{N} \to \mathbb{N} \\ F_k(0) &= 0 \\ F_k(n) &= n - F_k^{(k)}(n-1) \quad \text{otherwise} \end{aligned}$$

where $F_k^{(k)}$ is the k-th iterate $F_k \circ F_k \circ \cdots \circ F_k$.

In particular, $G = F_2$ and $H = F_3$.

This is suggested in Hofstadter's text, but does not appear explicitly.

What about F_0 and F_1 ?

- F_0 is a degenerate, non-recursive situation:

  $F_0(n) = 1$ when $n > 0$.

  Too different from the rest of the F_k family !

  We'll ignore it and only consider $k > 0$ from now on.

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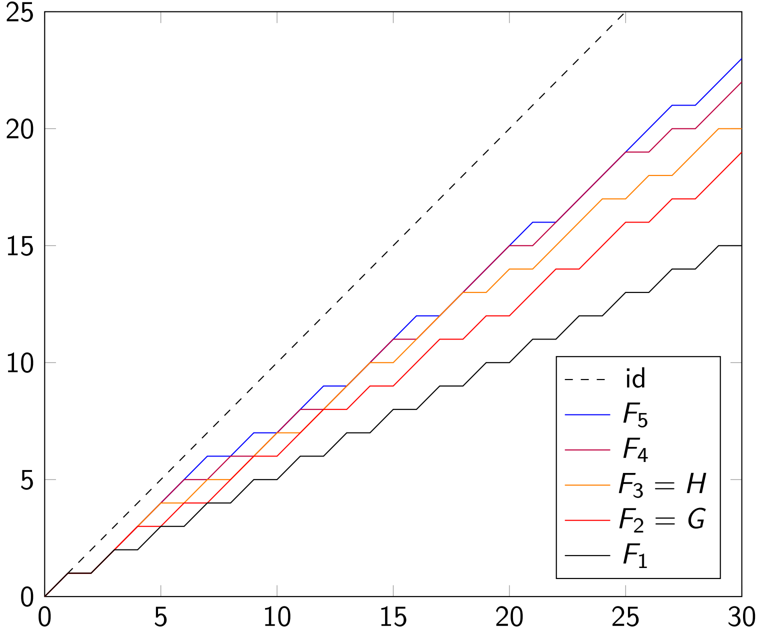
- F_1 is simply a division by 2 :

  $F_1(n) = n - F_1(n-1) = 1 + F_1(n-2)$ when $n \geq 2$.

  Actually $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$.

Plotting the first F_k



Some early properties of F_k

$$F_k(n) = n - F_k^{(k)}(n-1)$$

- Well-defined since $0 \leq F_k(n) \leq n$
- $F_k(0) = 0$, $F_k(1) = 1$ then $n/2 \leq F_k(n) < n$
- F_k is made of a mix of flats $(+0)$ and steps $(+1)$
- Hence each F_k is increasing, onto, but not one-to-one
- Never two flats in a row
- At most k steps in a row

Monotony of the F_k family

Pointwise order for functions : $f \leq h \iff \forall n, f(n) \leq h(n)$.

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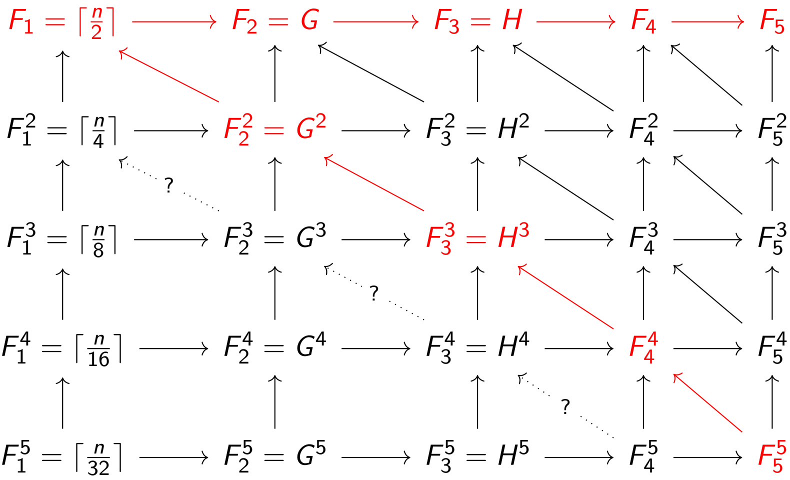
- Conjectured in 2018.
- First proof by Shuo Li (Nov 2023).
- Improved version by Wolfgang Steiner.
- Completely proved in Coq (as most of this talk).
- Proof ingredient : "detour" via some infinite morphic words.

More monotony

For $k > 0$ and $0 \leq j \leq k$:

$$
\begin{array}{ccccc}
F_k^j & \xrightarrow{\leq} & F_{k+1}^j & & \\
& \nwarrow^{\leq} & \uparrow{\scriptstyle \leq} & \nwarrow^{\leq} & \\
& & F_{k+1}^{j+1} & \xrightarrow[\leq]{} & F_{k+2}^{j+1}
\end{array}
$$

More monotony



Linear Equivalent

Let α_k be the positive root of $X^k + X - 1$.

Theorem: $\forall k > 0$, when $n \to \infty$ we have $F_k(n) = \alpha_k.n + o(n)$

Linear Equivalent

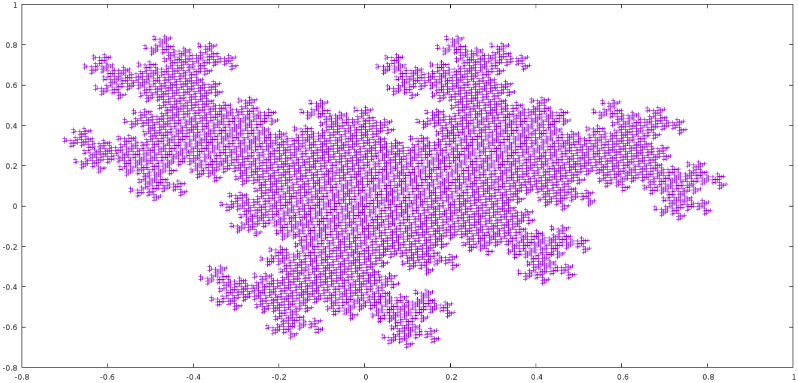
- $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$
- $G(n) = F_2(n) = \lfloor \alpha_2.(n+1) \rfloor$ with $\alpha_2 = \phi - 1 \approx 0.618...$

No more exact expression based on integral part of affine function. Instead:

- $H(n) = F_3(n) \in \lfloor \alpha_3.n \rfloor + \{0, 1\}$
- $F_4(n) \in \lfloor \alpha_4.n \rfloor + \{-1, 0, 1, 2\}$
- For $k \geq 5$, $F_k(n) - \alpha_k.n$ is no longer bounded.

Rauzy Fractal

Let $\delta(n) = F_3(n) - \alpha_3.n$. Then plotting $(\delta(i), \delta(F_3(i)))$ leads to this Rauzy fractal



Quiz !

Let $k > 0$. We say that a set of integers S is k-sparse if two distinct elements of S are always separated by at least k. How many k-sparse subsets of $\{1..n\}$ could you form ?

Generalized Fibonacci

For $k > 0$:

$$\begin{cases} A_{k,n} &= n + 1 & \text{when } n \leq k \\ A_{k,n} &= A_{k,n-1} + A_{k,n-k} & \text{when } n \geq k \end{cases}$$

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- $A_{1,n}$: 1 2 4 8 16 32 64 128 256 512 ... (Powers of 2)
- $A_{2,n}$: 1 2 3 5 8 13 21 34 55 89 ... (Fibonacci)
- $A_{3,n}$: 1 2 3 4 6 9 13 19 28 41 ... (Narayana's Cows)
- $A_{4,n}$: 1 2 3 4 5 7 10 14 19 26 ...

Zeckendorf decomposition

Let $k > 0$.

Theorem (Zeckendorf): all natural number can be written as a sum of $A_{k,i}$ numbers. This decomposition is unique when its indices i form a k-sparse set.

Theorem: F_k is a right shift for such a decomposition : $F_k(\Sigma A_{k,i}) = \Sigma A_{k,i-1}$ (with the convention $A_{k,0-1} = A_{k,0} = 1$)

NB: This shifted decomposition might not be k-sparse anymore

Key property : F_k is "flat" at n iff the decomposition of n contains $A_{k,0} = 1$.

More generally, $F_k^{(j)}$ is "flat" at n iff $j > rank(n)$ where the rank of n is the smallest index in the decomposition of n.

A letter substitution

Let $k > 0$. We use $\mathcal{A} = [1..k]$ as alphabet.

$$\begin{aligned} \mathcal{A} &\to \mathcal{A}^* \\ \tau_k(n) &= (n+1) && \text{pour } n < k \\ \tau_k(k) &= k.1 \end{aligned}$$

Starting from letter k, this generates an infinite word x_k (this word is said *morphic*).

For instance
$x_3 = 312331312312331233131233131231233131231 2\cdots$

Recursive equation on words

x_k is the limit of $\tau_k^n(k)$ when $n \to \infty$

It is also the limit of the following prefixes $M_{k,n}$:

- $M_{k,n} = k.0...(n-1)$ when $n \leq k$
- $M_{k,n} = M_{k,n-1}.M_{k,n-k}$ when $k \leq n$

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Note: $|M_{k,n}| = A_{k,n}$

Link with F_k

The n-th letter lettre $x_k[n]$ of the infinite word x_k is $\min(1{+}rank(n), k)$.

In particular this letter is 1 iff $F_k(n) = F_k(n+1)$

The count of letter 1 in x_k between 0 and $n-1$ is $n - F_k(n)$.

More generally, counting letters above p gives $F_k^{(p)}$. In particular the count of letter k is $F_k^{(k-1)}$.

No time today for:

- F_k admits a right adjoint (Galois connection), and this function behave as a left shift on the previous decompositions.
- A variant of F_k is already known to be a more conventional right shift on these decompositions (Meek & van Rees, 1981).
- An algebraic expression for $A_{k,n}$ fully based on the roots of $X^k - X^{k-1} - 1$.
- ...

Thank you for your attention

Coq Development : https://github.com/letouzey/hofstadter_g